Solutions - Test - II - 2017-18
Subject : Mathematics
Class : XII

Section - A

1) given: $\int_0^a \frac{dx}{4+x^2} = \frac{\pi}{8}$

$\Rightarrow \left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^a = \frac{\pi}{8}$

$\Rightarrow \frac{1}{\cancel{2}}\tan^{-1}\left(\frac{a}{2}\right) = \frac{\pi}{\cancel{8}\,4}$

$\Rightarrow \frac{a}{2} = \tan(\pi/4) = 1$

$\Rightarrow \boxed{a=2}$ Ans

2) If $f(x) = \int_0^x \frac{te^t}{(1+t)^2}\,dt$

$\Rightarrow f'(x) = \left[\frac{te^t}{(1+t)^2}\right]_0^x = \frac{xe^x}{(1+x)^2}$ Ans

$\left(\because f(x) = \int_0^x f'(t)\,dt\right)$

Section - B

3) Given curve is: $y = x^3 - 11x + 5$

$\Rightarrow \frac{dy}{dx} = 3x^2 - 11$

Let the pt of contact be (x_1, y_1)

$\Rightarrow m_T = \left.\frac{dy}{dx}\right]_{(x_1,y_1)} = 3x_1^2 - 11$

Given tangent is: $y = x - 11$

$\Rightarrow m_T = 1$

$\therefore 3x_1^2 - 11 = 1 \Rightarrow 3x_1^2 = 12$

$\Rightarrow x_1^2 = 4 \Rightarrow x_1 = \pm 2$

When $x_1 = 2$

$$y_1 = 2^3 - 11(2) + 5$$
$$= -9$$

When $x_1 = -2$

$$y_1 = (-2)^3 - 11(-2) + 5$$
$$= -8 + 22 + 5$$
$$= 19$$

But $(-2, 19)$ does not satisfy the eqn of tangent $(y = x - 11)$!

$\therefore$ Reqd point is $(2, -9)$ Ans.

4) $V = \frac{4}{3}\pi r^3$ → radius of spherical diamond

Volume of the spherical diamond

We know: $\Delta V = \frac{dV}{dr} \Delta r$ (approx)

$$\Rightarrow \Delta V = \frac{4\pi}{\cancel{3}} \times \cancel{3} r^2 (0.04)$$

$$= 4 \times \frac{22}{\cancel{7}} \times (7)^{\cancel{2}} \times (0.04)$$

$$= 24.64 \text{ cm}^3 \text{ (approx.)}$$

$\therefore$ Loss to the buyer $= 24.64 \times 1000$
$= ₹\ 24640$

My advice to the buyer:
Before buying the diamond get the measurement done carefully so as to minimise error in measurement as it is leading to a huge amount of loss.

5) Let $I = \int \frac{dx}{x(x^3+8)} = \int \frac{dx}{x^4(1+8x^{-3})}$

$$= \int \frac{x^{-4}dx}{(1+8x^{-3})}$$

Let $1 + 8x^{-3} = t$

$$\Rightarrow -3 \times 8\, x^{-4} dx = dt$$

$$\Rightarrow x^{-4} dx = \frac{-1}{24} dt$$

$$\therefore I = \frac{-1}{24}\int \frac{dt}{t}$$

$$\Rightarrow I = \frac{-1}{24}\log|t| + C$$

$$\Rightarrow I = \frac{-1}{24}\log|1+8x^{-3}| + C \quad \text{Ans}$$

6) Let $I = \int \frac{e^x\,dx}{\sqrt{5-4e^x-e^{2x}}}$

Let $e^x = t \Rightarrow e^x\,dx = dt$

$$\therefore I = \int \frac{dt}{\sqrt{5-4t-t^2}}$$

$$\Rightarrow I = \int \frac{dt}{\sqrt{-(t^2+4t-5 \underset{+2^2-2^2}{})}}$$

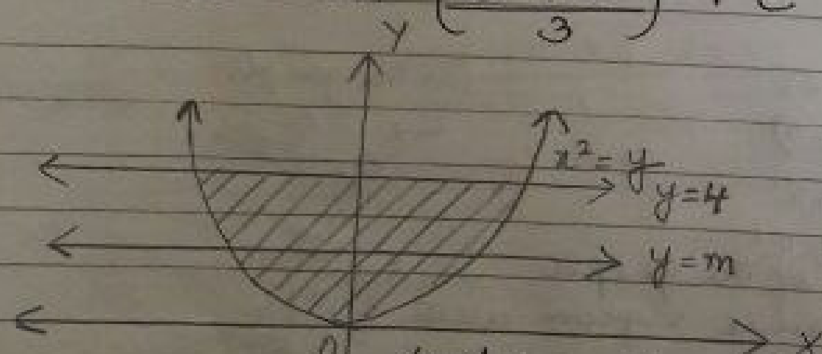
$$\Rightarrow I = \int \frac{dt}{\sqrt{-\{(t+2)^2-(3)^2\}}}$$

$$\Rightarrow I = \int \frac{dt}{\sqrt{(3)^2-(t+2)^2}}$$

$$\Rightarrow I = \sin^{-1}\left(\frac{t+2}{3}\right) + C$$

$$\Rightarrow I = \sin^{-1}\left(\frac{e^x+2}{3}\right) + C \quad \text{Ans.}$$

7)



Land is equally divide between daughter son

$$\Rightarrow \cancel{2}\int_{y=m}^{y=4}\sqrt{y}\,dy = \cancel{2}\int_0^m \sqrt{y}\,dy$$

$$\Rightarrow \frac{\cancel{2}}{\cancel{3}}\left(y^{3/2}\right)_m^4 = \frac{\cancel{2}}{\cancel{3}}\left(y^{3/2}\right)_0^m$$

$$\Rightarrow 4^{3/2} - m^{3/2} = m^{3/2} - 0$$

$$\Rightarrow 4^{3/2} = 2m^{3/2}$$

$$\Rightarrow \frac{(2^{\cancel{2}})^{3/\cancel{2}}}{2} = m^{3/2}$$

$$\Rightarrow 4 = m^{3/2}$$

$$\Rightarrow \boxed{m = 4^{2/3}}$$

The question promotes gender equality - Daughter being treated at par with sons.
Also it reflects use of integration in real life

Section - C

8) Let x kg of food I & y kg of food II be mixed.

Min $Z = 5x + 7y$

subject to
$$2x + y \geq 8$$
$$x + 2y \geq 10$$
$$x, y \geq 0$$

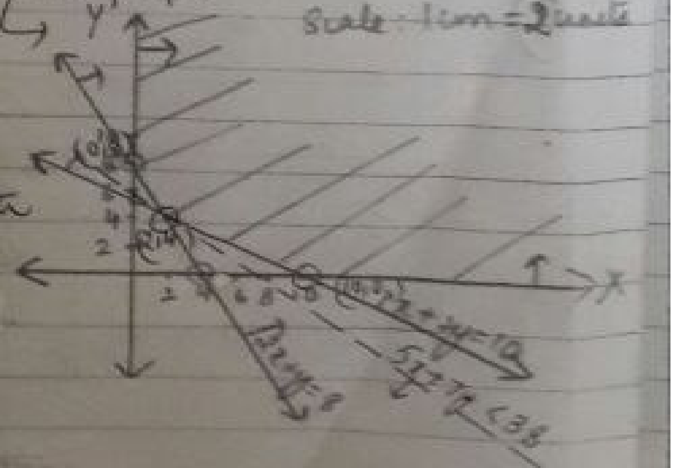
$2x+y=8 \; \langle \; (0,8), \; (4,0)$

$(2,4)$

$x+2y=10 \; \langle \; (0,5), \; (10,0)$

The shaded part of the graph represents the feasible region with cornerpoints $(0,8)$, $(2,4)$ & $(10,0)$

On the graph



Here $Z = 5x + 7y$

$\therefore Z(0,8) = ₹ 56$

$Z(2,4) = ₹ 38$ $\longrightarrow$ To confirm Min $Z = ₹ 38$

$Z(10,0) = ₹ 50$ we sketch $5x + 7y < 38$ (As the feasible region is unbounded)

From the graph it is clear that $5x + 7y < 38$ has no points common with the feasible region

$\Rightarrow$ Min $Z = ₹ 38$ Ans

9) Let $I = \int_{-\pi}^{\pi} \frac{2x(1+\sin x)dx}{1+\cos^2 x} = \int_{-\pi}^{\pi} \frac{2x\,dx}{1+\cos^2 x} + \int_{-\pi}^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$

neither even nor odd func

Let $f(x) = \frac{2x}{1+\cos^2 x}$

$\therefore f(-x) = \frac{-2x}{1+\cos^2 x} = -f(x)$

$\Rightarrow \int_{-\pi}^{\pi} \frac{2x}{1+\cos^2 x}dx = 0$

Let $g(x) = \frac{2x\sin x}{1+\cos^2 x}$

$\therefore g(-x) = g(x)$

$\therefore I = 0 + 2\int_0^{\pi} \frac{2x\sin x}{1+\cos^2 x}dx$

$\Rightarrow I = 4\int_0^{\pi} \frac{x\sin x\,dx}{1+\cos^2 x}$ — (1)

$\Rightarrow I = 4\int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}dx$

$\Rightarrow I = 4\int_0^{\pi} \frac{(\pi - x)\sin x}{1+\cos^2 x}dx$ — (2)

(1) + (2) gives:

$2I = 4\pi \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}dx$

Let $\cos x = t \Rightarrow -\sin x\,dx = dt$

when $x = 0,\ t = 1$

$x = \pi,\ t = -1$

$$\Rightarrow I = \frac{-4\pi}{2}\int_{1}^{-1}\frac{dt}{1+t^2}$$

$$\Rightarrow I = -2\pi\left[\tan^{-1}t\right]_{1}^{-1}$$

$$\Rightarrow I = -2\pi\left[\tan^{-1}(-1) - \tan^{-1}(1)\right]$$

$$\Rightarrow I = -2\pi\left[-\tan^{-1}(1) - \tan^{-1}(1)\right]$$

$$\Rightarrow I = -2\pi\left[-2\tan^{-1}(1)\right]$$

$$\Rightarrow I = 4\pi\tan^{-1}(1)$$

$$\Rightarrow I = \cancel{4}\pi\left(\frac{\pi}{\cancel{4}}\right)$$

$$\Rightarrow \boxed{I = \pi^2} \text{ Ans}$$

OR

$$I = \int_{\pi/6}^{\pi/3}\frac{\cos x + \sin x}{\sqrt{\sin 2x}}\,dx$$

Let $\sin x - \cos x = t$

$$\Rightarrow (\cos x + \sin x)\,dx = dt$$

$$\therefore (\sin x - \cos x)^2 = t^2$$

$$\Rightarrow 1 - \sin 2x = t^2$$

When $x = \frac{\pi}{3}$, $t = \frac{\sqrt{3}-1}{2}$

$x = \pi/6$, $t = \frac{1-\sqrt{3}}{2}$

$$\therefore I = \int_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}}\frac{dt}{\sqrt{1-t^2}}$$

$$\Rightarrow I = \left(\sin^{-1}t\right)_{\frac{1-\sqrt{3}}{2}}^{\frac{\sqrt{3}-1}{2}} = \sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) - \sin^{-1}\left(\frac{1-\sqrt{3}}{2}\right)$$

$$\Rightarrow I = 2\sin^{-1}\left(\frac{\sqrt{3}-1}{2}\right) \text{ Ans}$$

10) $y^2 = 2x$ —(1)

↳ is a parabola with vertex (0,0) & opening right

$x^2+y^2 = 4x$

$\Rightarrow x^2+y^2-4x = 0$

$\Rightarrow x^2-4x+2^2-2^2+y^2=0$

$\Rightarrow (x-2)^2+y^2 = 2^2$ —(2)

↳ is a circle with centre (2,0) & r = 2 units

Pt. of intersection of (1) & (2):

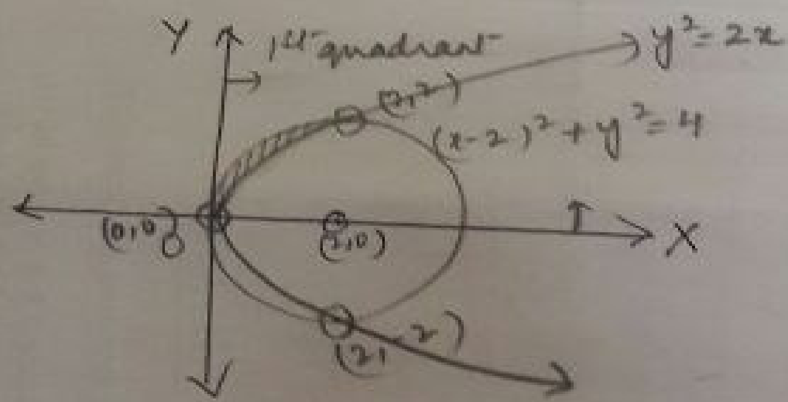
$x^2+2x = 4x \Rightarrow x^2-2x=0$

$\Rightarrow x(x-2)=0$

$\Rightarrow x = 0, 2$

$y = 0 \quad \pm 2$

i.e (0,0) & (2, ±2)



The shaded part of the figure represents the required area & Reqd Area $= \int_0^2 \left(\sqrt{4-(x-2)^2} - \sqrt{2x}\right)dx$

$= \int_0^2 \sqrt{4-(x-2)^2}\,dx - \sqrt{2}\int_0^2 x^{1/2}dx$

$= \left(\frac{x-2}{2}\sqrt{4-(x-2)^2} + \frac{4}{2}\sin^{-1}\left(\frac{x-2}{2}\right)\right)_0^2 - \sqrt{2}\times\frac{2}{3}\left(x^{3/2}\right)_0^2$

$= 2\left[\sin^{-1}0 - \sin^{-1}(-1)\right] + \frac{2\sqrt{2}}{3}\left(2^{3/2}\right)$

$= 2\left[\frac{\pi}{2}\right] + \frac{2\sqrt{2}}{3}\times 2\sqrt{2} = \left(\pi + \frac{8}{3}\right)$ sq. units

Ans

11) Let $S =$ Surface area of sphere + Surface area of cuboid

$$\Rightarrow S = \underbrace{4\pi r^2}_{r = \text{radius of sphere}} + 2\left(\frac{x}{3}\times x + x\times 2x + \frac{x}{3}\times 2x\right)$$

$$\Rightarrow S = 4\pi r^2 + 2\left(\frac{x^2}{3} + 2x^2 + \frac{2x^2}{3}\right)$$

$$\Rightarrow S = 4\pi r^2 + 2(3x^2)$$

$$\Rightarrow S = 4\pi r^2 + 6x^2 \quad \text{--- (1)}$$

& $V = \frac{4}{3}\pi r^3 + \frac{x}{3}\times x\times 2x$

volume of sphere + cuboid

$$\Rightarrow V = \frac{4}{3}\pi r^3 + \frac{2x^3}{3} \quad \text{--- (2)}$$

$$\Rightarrow V = \frac{4\pi}{3}r^3 + \frac{2}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{3/2}$$

$$\frac{dV}{dr} = \frac{4\pi}{\cancel{3}}\times \cancel{3}r^2 + \frac{\cancel{2}}{\cancel{3}}\times\frac{\cancel{3}}{\cancel{2}}\left(\frac{S - 4\pi r^2}{6}\right)^{1/2}\left(\frac{-\cancel{8}^{4}\pi r}{\cancel{6}3}\right)$$

$$\Rightarrow \frac{dV}{dr} = 4\pi r^2 - \frac{4\pi r}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{1/2}$$

$$\frac{dV}{dr} = 0 \Rightarrow \cancel{4\pi} r^{\cancel{2}} = \frac{\cancel{4\pi r}}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{1/2}$$

$$\Rightarrow 3r = \left(\frac{S - 4\pi r^2}{6}\right)^{1/2}$$

$$\Rightarrow 3r = \left(\frac{\cancel{6}x^2}{\cancel{6}}\right)^{1/2} \quad (\text{Using eq (1)})$$

$$\Rightarrow \boxed{3r = x}$$

$$\frac{d^2V}{dr^2} = 8\pi r - \left[\frac{4\pi}{3}\left(\frac{S - 4\pi r^2}{6}\right)^{1/2} + \frac{\cancel{4}^{2}\pi r}{3}\times\frac{1}{\cancel{2}}\left(\frac{S - 4\pi r^2}{6}\right)^{-1/2}\times\frac{-8\pi r}{\cancel{6}3}\right]$$

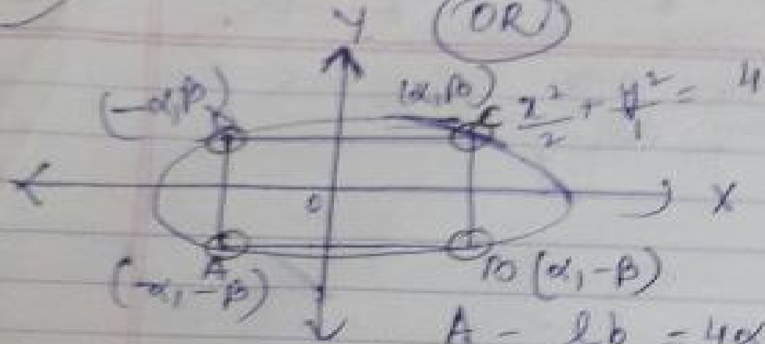
$$= 8\pi r - \frac{4\pi}{3}\sqrt{\frac{S - 4\pi r^2}{6}} + \frac{8\pi^2 r^2}{9}\left(\frac{S - 4\pi r^2}{6}\right)^{-1/2}$$

> 0 at the turning point

$\Rightarrow V$ is minimum & for min^m V, $\boxed{3r = x}$

11

OR



$l = 2\alpha$

$b = 2\beta$.

$A = lb = 4\alpha\beta$. — (1)

(α,β) lies on $\frac{x^2}{2}+\frac{y^2}{1}=4$

$y = \sqrt{4-\frac{x^2}{2}}$

$\Rightarrow y = \frac{1}{\sqrt{2}}\sqrt{8-x^2}$.

$\Rightarrow \beta = \frac{\sqrt{8-\alpha^2}}{\sqrt{2}}$

$A = 4\alpha\beta = \frac{\cancel{4}^{2\sqrt{2}}\alpha}{\cancel{\sqrt{2}}}\sqrt{8-\alpha^2}$

$\frac{dA}{d\alpha} = 2\sqrt{2}\left[\sqrt{8-\alpha^2} + \frac{\alpha}{\cancel{2}\sqrt{8-\alpha^2}}(-\cancel{2}\alpha)\right]$

$= 2\sqrt{2}\left[\frac{8-\alpha^2-\alpha^2}{\sqrt{8-\alpha^2}}\right] = \frac{2\sqrt{2}(8-2\alpha^2)}{\sqrt{8-\alpha^2}}$

$\frac{dA}{d\alpha} = 0 \Rightarrow 8 = 2\alpha^2$

$\Rightarrow 4 = \alpha^2 \Rightarrow \alpha = \pm 2$

when $\alpha = 2$; $\beta = \frac{\sqrt{8-4}}{\sqrt{2}} = \frac{2}{\sqrt{2}} = \sqrt{2}$

$\frac{d^2A}{d\alpha^2} = 2\sqrt{2}\left[\frac{\sqrt{8-\alpha^2}(-4\alpha) - \frac{(8-2\alpha^2)}{\cancel{2}\sqrt{8-\alpha^2}}(-\cancel{2}\alpha)}{(8-\alpha^2)}\right]$

< 0 for $\alpha = 2$

$\therefore$ A is greatest when $\alpha = 2$, $\beta = \sqrt{2}$.

$\Rightarrow$ sides of the inscribed rectangle with greatest area are $2\alpha, 2\beta$ i.e $4, 2\sqrt{2}$ units

i.e Length = 4 units & Breadth = $2\sqrt{2}$ units Ans