TEST - I - 2018-19 - class - XII - Sub - Mathematics

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Solutions :

1) $A = [a_{ij}]_{1\times 2} = [a_{11} \quad a_{12}]$

Here $a_{ij} = i - j^2$. $\therefore a_{11} = 1 - 1^2 = 0$

& $a_{12} = 1 - 2^2 = -3$.

$\therefore A = [0 \quad -3]$ Ans

2) $X^2 = (XY)^2$ $(\because X = XY \text{ given})$

$= XYXY$

$= XYY$ $(\because YX = Y \text{ given})$

$= XY$ $(\because X = XY \text{ given})$

$= X$ $(\because X = XY \text{ given})$

$\Rightarrow X^2 = X$. Hence Proved

3) $A = \begin{bmatrix} -1 & 1 \\ a & 1 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} -1 & a \\ 1 & 1 \end{bmatrix}$

$\therefore AA^T = \begin{bmatrix} -1 & 1 \\ a & 1 \end{bmatrix}\begin{bmatrix} -1 & a \\ 1 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & -a+1 \\ -a+1 & a^2+1 \end{bmatrix}$

AA^T is a scalar matrix $\Rightarrow 2 = a^2+1$ & $-a+1 = 0$

$\Rightarrow a^2 = 1$ & $a = 1$

$\Rightarrow a = \pm 1$ & $a = 1$

$\Rightarrow a = 1$ Ans.

4) $A - kI = \begin{bmatrix} -2 & -3 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$

$= \begin{bmatrix} -2-k & -3 \\ -1 & 2-k \end{bmatrix}$

$A - kI$ has no inverse $\Rightarrow |A - kI| = 0$

$\Rightarrow (-2-k)(2-k) - 3 = 0$

$\Rightarrow -4 + \cancel{2k} - \cancel{2k} + k^2 - 3 = 0$

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$\Rightarrow k^2 - 7 = 0$

$\Rightarrow k = \pm\sqrt{7}$ Ans.

5) $\tan(\cos^{-1}x) = \sin(\cot^{-1} 1/2)$

$\Rightarrow \tan\left(\tan^{-1}\frac{\sqrt{1-x^2}}{x}\right) = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$

$\Rightarrow \frac{\sqrt{1-x^2}}{x} = \frac{2}{\sqrt{5}}$

$\Rightarrow 5(1-x^2) = 4x^2$

$\Rightarrow 5 = 9x^2$

or $x = \pm\frac{\sqrt{5}}{3}$ Ans.

OR (Alt. Soln)

$\tan(\cos^{-1}x) = \sin\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)$

$\Rightarrow \tan(\cos^{-1}x) = \frac{2}{\sqrt{5}}$

$\Rightarrow \cos^{-1}x = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right)$

$\Rightarrow \cos^{-1}x = \cos^{-1}\left(\frac{\sqrt{5}}{3}\right)$

$\Rightarrow x = \frac{\sqrt{5}}{3}$ Ans.

6) Yes $*$ on Z^+ is a binary operation $\because a*b$ i.e $a+b \in Z^+$ $\forall a,b \in Z^+$

Let $e \in Z^+$ be the identity elt for $*$ on Z^+

$\Rightarrow a*e = e*a = a$

$\Rightarrow a+e = a$

$\Rightarrow e = 0 \notin Z^+$

$\therefore$ id elt does not exist for $*$ on Z^+

$\Rightarrow$ inverse also does not exist for $*$ on Z^+

7) $(x-y)\,e^{\frac{x}{x-y}} = a$

Taking log of b/s

$\log_e (x-y)\,e^{\frac{x}{x-y}} = \log_e a$

$\Rightarrow \log_e(x-y) + \left(\frac{x}{x-y}\right)\log_e e = \log_e a$

$\Rightarrow \log_e(x-y) + \frac{x}{x-y} = \log_e a$

diffn b/s w.r.t 'x' we get:

$$\frac{1}{x-y}\left(1-\frac{dy}{dx}\right) + \left(\frac{(x-y)1 - x\left(1-\frac{dy}{dx}\right)}{(x-y)^2}\right) = 0$$

$$\Rightarrow \frac{1}{x-y} + \frac{1}{x-y} - \frac{x}{(x-y)^2} = \frac{dy}{dx}\left[\frac{1}{x-y} - \frac{x}{(x-y)^2}\right]$$

$$\Rightarrow \frac{2}{x-y} - \frac{x}{(x-y)^2} = \frac{dy}{dx}\left[\frac{\cancel{x} - y - \cancel{x}}{(x-y)^2}\right]$$

$$\Rightarrow \frac{2x-2y-x}{\cancel{(x-y)^2}} = \frac{dy}{dx}\left(\frac{-y}{\cancel{(x-y)^2}}\right)$$

$$\Rightarrow x - 2y = -y\frac{dy}{dx}$$

$$\Rightarrow 2y - x = y\frac{dy}{dx}$$

$$\Rightarrow \boxed{y\frac{dy}{dx} + x = 2y} \quad \text{H.P.}$$

8) f is diffn at $x=1$

$\Rightarrow f$ is cont. at $x=1$

$\Rightarrow$ LHL $=$ RHL at $x=1$

$$\Rightarrow \lim_{x\to 1^-}(5ax^3 - 4bx) = \lim_{x\to 1^+}(3ax^2 + 2bx - 14)$$

$$\Rightarrow 5a - 4b = 3a + 2b - 14$$

$$\Rightarrow 2a - 6b = -14$$

$$\Rightarrow \boxed{3b - a = 7} \text{ — (1)}$$

Also LHD $=$ RHD at $x=1$

$$\Rightarrow \lim_{h\to 0}\frac{f(1-h)-f(1)}{-h} = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left[5a(1-h)^3 - 4b(1-h)\right] - (5a-4b)}{-h} = \lim_{h\to 0}\frac{\left[3a(1+h)^2 + 2b(1+h) - 14\right] - (5a-4b)}{h}$$

$$\Rightarrow \lim_{h\to 0}\frac{\left[5a(\cancel{1} - h^3 - 3h + 3h^2) - \cancel{4b} + 4bh\right] - \cancel{5a} + \cancel{4b}}{-h} = \lim_{h\to 0}\frac{\left[3a(1+h^2+2h) + 2b + 2bh - 14\right] - 5a + 4b}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{h[-5ah^2 - 15a + 15ah + 4b]}{-h} = \lim_{h\to 0} \frac{\left[3ah^2 + 6ah + 2bh - \cancel{2a} + \cancel{6b} - \cancel{14}\right]}{h}$$

$$\Rightarrow -\lim_{h\to 0} [-5ah^2 - 15a + 15ah + 4b] = \lim_{h\to 0} \frac{\cancel{h}[3ah + 6a + 2b]}{\cancel{h}}$$

$$\Rightarrow \quad 15a - 4b = 6a + 2b$$

$$\Rightarrow \quad 9a = 6b \Rightarrow \boxed{a = 2/3\, b} \quad -②$$

Using ② in eq ① we get:

$$3b - \frac{2}{3}b = 7$$

$$\Rightarrow \frac{\cancel{7}b}{3} = \cancel{7}$$

$$\left.\begin{aligned} \Rightarrow \quad & b = 3 \\ \Rightarrow \quad & a = 2 \end{aligned}\right\} \text{Ans.}$$

9) $$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta} \quad — ①$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(-a\sin\theta + b\cos\theta)(-a\sin\theta + b\cos\theta) - (a\cos\theta + b\sin\theta)(-a\cos\theta - b\sin\theta)}{(-a\sin\theta + b\cos\theta)^2} \times \frac{d\theta}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{(b\cos\theta - a\sin\theta)^2 + (a\cos\theta + b\sin\theta)^2}{(-a\sin\theta + b\cos\theta)^2} \times \frac{1}{-a\sin\theta + b\cos\theta}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{b^2\cos^2\theta + a^2\sin^2\theta \cancel{- 2ab\cos\theta\sin\theta} + a^2\cos^2\theta + b^2\sin^2\theta \cancel{+ 2ab\sin\theta\cos\theta}}{(-a\sin\theta + b\cos\theta)^3}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{a^2 + b^2}{(-a\sin\theta + b\cos\theta)^3} \quad -②$$

$$L.H.S = y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y$$

$$= (a\sin\theta - b\cos\theta)^2 \times \frac{(a^2+b^2)}{(-a\sin\theta + b\cos\theta)^3} - \frac{(a\cos\theta + b\sin\theta)^2}{-a\sin\theta + b\cos\theta} + (a\sin\theta - b\cos\theta)$$

$$\Rightarrow LHS = \frac{a^2+b^2}{-(+a\sin\theta - b\cos\theta)} + \frac{(a\cos\theta + b\sin\theta)^2}{a\sin\theta - b\cos\theta} + (a\sin\theta - b\cos\theta)$$

$$= \frac{-(a^2+b^2) + (a\cos\theta + b\sin\theta)^2 + (a\sin\theta - b\cos\theta)^2}{(a\sin\theta - b\cos\theta)}$$

$$= \frac{-(a^2+b^2) + (a^2\cos^2\theta + b^2\sin^2\theta + 2ab\sin\theta\cos\theta) + (a\sin\theta - b\cos\theta)^2}{(-a\sin\theta + b\cos\theta)}$$

$$= \frac{-(a^2+b^2) + (a^2\cos^2\theta + b^2\sin^2\theta + \cancel{2ab\sin\theta\cos\theta}) + a^2\sin^2\theta + b^2\cos^2\theta - \cancel{2ab\sin\theta\cos\theta}}{(-a\sin\theta + b\cos\theta)}$$

$$= \frac{-a^2 - b^2 + a^2 + b^2}{-a\sin\theta + b\cos\theta} = 0 \quad \text{Ans}$$

Alt Soln Q 9.

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{a\cos\theta + b\sin\theta}{-a\sin\theta + b\cos\theta} = \frac{x}{-y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-y(1) - x\left(-\frac{dy}{dx}\right)}{y^2}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-1}{y} + \frac{x}{y^2}\frac{dy}{dx}$$

$$\Rightarrow \boxed{y^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0} \quad \text{Ans.}$$

10) $\Delta(x) = \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x(x-1) & x(x-1)(x-2) & x(x^2-1) \end{vmatrix}$

Taking $(x-1)$ common from R_3:

$\Delta(x) = (x-1) \begin{vmatrix} 1 & x & x+1 \\ 2x & x(x-1) & x(x+1) \\ 3x & x(x-2) & x(x+1) \end{vmatrix}$

Taking x common from C_2 & $(x+1)$ common from C_3

$\Delta(x) = (x-1)\, x\, (x+1) \begin{vmatrix} 1 & 1 & 1 \\ 2x & x-1 & x \\ 3x & x-2 & x \end{vmatrix}$

$C_3 \to C_3 - C_1$ & $C_2 \to C_2 - C_1$

$\Delta(x) = x(x^2-1) \begin{vmatrix} 1 & 0 & 0 \\ 2x & -1-x & -x \\ 3x & -2-2x & -2x \end{vmatrix}$

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Exp along R_1 :

$$\Delta(x) = x(x^2-1) \begin{vmatrix} -1-x & -x \\ -2-2x & -2x \end{vmatrix}$$

Taking x common from C_2 :

$$\Delta(x) = x^2(x^2-1) \begin{vmatrix} -1-x & -1 \\ -2-2x & -2 \end{vmatrix}$$

$$\Rightarrow \Delta(x) = x^2(x^2-1) \{-2(-1-x) - (-1)(-2-2x)\}$$
$$\Rightarrow \Delta(x) = x^2(x^2-1) \{\cancel{2} + \cancel{2x} - \cancel{2} - \cancel{2x}\}$$
$$\Rightarrow \Delta(x) = 0$$
$$\therefore \Delta(10) = 0 \text{ Ans.}$$

OR

The given system of eqns can be expressed as:

$AX = B$ where $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 1 & 1 \\ 1 & -3 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ & $B = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$

Here $|A| = 1(1+3) - 2(-1-1) + 1(3-1)$
$= 4 + 4 + 2$
$= 10 \neq 0 \Rightarrow$ given system of eqns is consistent with a unique soln given by : $\boxed{X = A^{-1}B}$

To find A^{-1}

$C_{11} = 4 \quad C_{12} = 2 \quad C_{13} = 2$
$C_{21} = -5 \quad C_{22} = 0 \quad C_{23} = 5$
$C_{31} = 1 \quad C_{32} = -2 \quad C_{33} = 3.$

$$\therefore \text{Adj}A = C^T = \begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$$

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$\therefore A^{-1} = \dfrac{Adj A}{|A|} = \dfrac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}$

$\Rightarrow X = A^{-1} B$

or $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 4 & -5 & 1 \\ 2 & 0 & -2 \\ 2 & 5 & 3 \end{bmatrix}\begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \dfrac{1}{10}\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$

$\Rightarrow x = 2, y = 0$ & $z = 2$ Ans.

11) ~~Let~~ Here $f(x) = x + \dfrac{1}{x}$

$\Rightarrow f(x) = \dfrac{x^2+1}{x}$; $x \in [1, \infty)$

For One-One:

Let $f(x_1) = f(x_2)$; $x_1, x_2 \in [1, \infty)$

$\Rightarrow \dfrac{x_1^2+1}{x_1} = \dfrac{x_2^2+1}{x_2}$

$\Rightarrow x_1^2 + x_2 = x_2^2 + x_1$

$\Rightarrow x_1^2 - x_2^2 = x_1 - x_2$

$\Rightarrow (x_1 - x_2)(x_1 + x_2) = (x_1 - x_2)$

$\Rightarrow (x_1 - x_2)(x_1 + x_2 - 1) = 0.$

$\Rightarrow x_1 - x_2 = 0$ $\quad (\because x_1 + x_2 - 1 \neq 0$ for $x_1, x_2 \in [1, \infty))$

$\Rightarrow x_1 = x_2$

$\Rightarrow f$ is one-one.

For Onto: Let $y \in$ codomain f

i.e $y \in [2, \infty)$

$\Rightarrow y = f(x)$

$\Rightarrow \quad y = \dfrac{x^2+1}{x}$

$\Rightarrow \quad xy = x^2+1$

$\Rightarrow \quad x^2 - xy + 1 = 0$

$\Rightarrow \quad x = \dfrac{y \pm \sqrt{y^2-4}}{2}$

$$x = \frac{y+\sqrt{y^2-4}}{2} \in [1,\infty)$$

$\Rightarrow \forall y \in [2,\infty) \ \exists \ x \in [1,\infty)$

i.e $\forall$ elt in the codomain $\exists$ a pre-image in the domain

$\Rightarrow$ f is onto

$\because f$ is one-one onto $\Rightarrow f^{-1}$ exists & is defined as:

$$f^{-1} : [2,\infty) \rightarrow [1,\infty)$$

s.t. $f^{-1}(x) = \dfrac{x+\sqrt{x^2-4}}{2}$ Ans.

OR

i) Reflexivity: $(a,b)\ R\ (a,b) \quad \because ab = ba \ \forall \ (a,b) \in A \times A$

$\Rightarrow$ R is reflexive

ii) Symmetry: Let $(a,b)\ R(c,d)$; $a,b,c,d \in A$

$\Rightarrow \quad ad = bc$

$\Rightarrow \quad da = cb$

$\Rightarrow \quad cb = da$

$\Rightarrow (c,d)\ R\ (a,b)$

$\Rightarrow$ R is symmetric

(iii) <u>Transitivity:</u> Let $(a,b) R (c,d)$ & $(c,d) R (e,f)$; $a,b,c,d,e,f \in A$

$\Rightarrow ad = bc$ & $cf = de$

$\Rightarrow a\cancel{d}\cancel{c}f = b\cancel{c}\cancel{d}e$

$\Rightarrow af = be$

$\Rightarrow (a,b) R (e,f)$

Since R is reflexive, symmetric & Transitive

$\Rightarrow$ R is an equivalence relation.

<u>For Equivalence class of (3,2):</u>

Let $(x,y) R (3,2)$

$\Rightarrow 2x = 3y$

$\Rightarrow y = \frac{2}{3}x$

$\therefore$ Equivalence class of (3,2) is:

$[(3,2), (6,4), (9,6), (12,8), (15,10), (18,12)]$

Ans.