Solutions - Class XII - 2018-19 - Half Yearly Exam - Sub - Mathematics

Section - A

1) $\frac{\pi}{2} - \cot^{-1}x + \frac{\pi}{2} - \cot^{-1}y = \frac{4\pi}{5}$

$\Rightarrow \pi - \frac{4\pi}{5} = \cot^{-1}x + \cot^{-1}y$

$\Rightarrow \pi/5 = \cot^{-1}x + \cot^{-1}y$ Ans.

2) $|2AA^{-1}| = 2|A||A^{-1}|$

$= 2|I|$

$= 2^3(1) = 2^3 = 8$ Ans.

3) Given : $B = B'$

$\therefore (AB'A')' = (A')'(B')'A'$

$= ABA'$

$= AB'A'$

$\Rightarrow AB'A'$ is symmetric

4) $\cos\left[\pi - \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right]$

$= \cos\left[\pi - \cancel{\frac{\pi}{6}} + \cancel{\frac{\pi}{6}}\right]$

$= -1$ Ans

Section - B

5) $\cos^{-1}\left(-\left(\frac{1-x^2}{1+x^2}\right)\right) - \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \pi - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) - \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \frac{2\pi}{3}$

$\Rightarrow \pi - 2\tan^{-1}x - 2\tan^{-1}x = \frac{2\pi}{3}$

$\Rightarrow \pi - 4\tan^{-1}x = \frac{2\pi}{3}$

$\Rightarrow \pi - \frac{2\pi}{3} = 4\tan^{-1}x$

$\Rightarrow \tan^{-1}x = \frac{\pi}{12}$

$\Rightarrow x = \tan\left(\frac{\pi}{12}\right)$ Ans.

6) $\text{LHS} = \begin{vmatrix} 0 & -3 & -4 \\ 3 & 0 & -7 \\ 4 & 7 & 0 \end{vmatrix}$

$C_1 \to C_1 + C_3$ & $C_2 \to C_2 + C_3$

$\Rightarrow \text{LHS} = \begin{vmatrix} -4 & -7 & -4 \\ -4 & -7 & -7 \\ 4 & 7 & 0 \end{vmatrix}$

$= 0$ ($\because C_1$ & C_2 are proportional)

$=$ R.H.S. Hence Proved

7) $\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{2\times 1} A = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}_{2\times 2}$ given

$\Rightarrow$ A is a matrix of order 1×2

Let $A = \begin{bmatrix} a & b \end{bmatrix}$

$\therefore \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} a & b \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a & b \\ 3a & 3b \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -3 & 6 \end{bmatrix}$

$\Rightarrow a = -1$ & $b = 2$

$\therefore A = \begin{bmatrix} -1 & 2 \end{bmatrix}$ Ans

8) $\frac{dy}{dx} = e^{f(x)} f'(x)$

$\therefore \left.\frac{dy}{dx}\right]_{x=0} = e^{f(0)} \cdot f'(0)$

$= e^2 \cdot 2$

$= 2e^2$ Ans

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9) $x = \sin^{-1}\left(\frac{2t}{1+t^2}\right)$ and $y = \tan^{-1}\left(\frac{2t}{1-t^2}\right)$

$\Rightarrow x = 2\tan^{-1}t$ & $y = 2\tan^{-1}t$

$$\therefore \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{2}{1+t^2}}{\frac{2}{1+t^2}} = 1 \text{ Ans.}$$

10) $\int \sqrt{(9)^2 - (7x)^2}\, dx$

$$= \frac{x}{2}\sqrt{81 - 49x^2} + \frac{81}{2\times 7}\sin^{-1}\left(\frac{7x}{9}\right) + C$$

$$= \frac{x}{2}\sqrt{81 - 49x^2} + \frac{81}{14}\sin^{-1}\left(\frac{7x}{9}\right) + C$$ Ans.

11) $\int \frac{\sin(x-a)}{\sin(x-b)}\, dx$

$$= \int \frac{\sin(x - b + b - a)}{\sin(x-b)}\, dx$$

$$= \int \frac{\sin[(x-b) - (a-b)]}{\sin(x-b)}\, dx$$

$$= \int \frac{\sin(x-b)\cos(a-b) - \cos(x-b)\sin(a-b)}{\sin(x-b)}\, dx$$

$$= \cos(a-b)\int dx - \sin(a-b)\int \cot(x-b)\, dx$$

$$= x\cos(a-b) - \sin(a-b)\log|\sin(x-b)| + C$$

Ans

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12) $f: R \to R$ s.t. $f(x) = e^x$

$f(x)$ is not onto because elements belonging to the set $(-\infty, 0]$ in the co-domain R do not have a pre-image in domain R.

f becomes onto if $f: R \to R^+$

i.e. if codomain f is modified from R to R^+ the set of positive real numbers.

Section - C

13) At $x = -2$

$L.H.L = \lim_{x \to -2^-} (2x+3)$
$= 2(-2)+3$
$= -1$

$R.H.L = \lim_{x \to -2^+} (x+1)$
$= -2+1$
$= -1$

$f(-2) = -2+1 = -1$

$LHL = R.H.L = f(-2) \Rightarrow f$ is continuous at $x = -2$

Differentiability at $x = -2$

$$L.H.D = \lim_{h \to 0} \frac{f(-2-h) - f(-2)}{-h}$$

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$$= \lim_{h \to 0} \frac{\left(2(-2-h)+3\right) - (-1)}{-h}$$

$$= \lim_{h \to 0} \frac{(-\cancel{4}-2h+\cancel{3})+\cancel{1}}{-h}$$

$$= \lim_{h \to 0} (2) = 2$$

$$R.H.D = \lim_{h \to 0} \frac{f(-2+h)-f(-2)}{h}$$

$$= \lim_{h \to 0} \frac{\left((-2+h)+1\right)-(-1)}{h}$$

$$= \lim_{h \to 0} \frac{(-\cancel{1}+h)+\cancel{1}}{h}$$

$$= 1 \neq L.H.D$$

$\Rightarrow$ Not differentiable at $x = -2$.

$\therefore$ given function is continuous but not differentiable at $x = -2$

14) $y = x^3 \log(1/x)$

$$= x^3(\log 1 - \log x)$$

$$\Rightarrow y = -x^3 \log x \quad -(1)$$

$$\therefore \frac{dy}{dx} = -3x^2 \log x - \frac{x^3}{x}.$$

$$\Rightarrow \frac{dy}{dx} = -3x^2 \log x - x^2. \quad -(2)$$

$$\therefore \frac{d^2y}{dx^2} = -6x \log x - 3x^2 \cdot \frac{1}{x} - 2x.$$

$$= -6x \log x - 5x. \quad -(3)$$

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$$L.H.S = x\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3x^2$$

$$= x(-6x\log x - 5x) - 2(-3x^2\log x - x^2) + 3x^2$$

$$= \cancel{-6x^2\log x} - \cancel{5x^2} + \cancel{6x^2\log x} + \cancel{2x^2} + \cancel{3x^2}$$

$$= 0 = R.H.S.$$

Hence Proved.

15) $y = \sin^{-1}\left[\frac{\sqrt{1+x^m} + \sqrt{1-x^m}}{2}\right]$

Let $x^m = \cos\theta$.

$$\therefore y = \sin^{-1}\left[\frac{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}{2}\right]$$

$$\Rightarrow y = \sin^{-1}\left[\frac{\sqrt{2}\cos\theta/_2 + \sqrt{2}\sin\theta/_2}{2}\right]$$

$$\Rightarrow y = \sin^{-1}\left[\frac{1}{\sqrt{2}}\cos\frac{\theta}{2} + \frac{1}{\sqrt{2}}\sin\frac{\theta}{2}\right]$$

$$\Rightarrow y = \sin^{-1}\left[\sin\frac{\pi}{4}\cos\frac{\theta}{2} + \cos\frac{\pi}{4}\cos\frac{\theta}{2}\right]$$

$$\Rightarrow y = \sin^{-1}\left[\sin\left(\frac{\pi}{4} + \frac{\theta}{2}\right)\right]$$

$$\Rightarrow y = \frac{\pi}{4} + \frac{\theta}{2}$$

$$\Rightarrow y = \frac{\pi}{4} + \frac{1}{2}\cos^{-1}x^m.$$

$$\therefore \frac{dy}{dx} = \frac{1}{2} \times \frac{-1}{\sqrt{1-(x^m)^2}} \cdot mx^{m-1}$$

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$\Rightarrow \dfrac{dy}{dx} = -\dfrac{mx^{m-1}}{2\sqrt{1-x^{2m}}}$ Ans.

16) Let $y = (5x)^{3\cos 2x} + \dfrac{1}{\sqrt{x^2+2x}}$

or $y = u + v$ where $u = (5x)^{3\cos 2x}$

& $v = \dfrac{1}{\sqrt{x^2+2x}}$

$\therefore \dfrac{dy}{dx} = \dfrac{du}{dx} + \dfrac{dv}{dx}$ — (1)

For $\dfrac{du}{dx}$: $u = (5x)^{3\cos 2x}$

$\Rightarrow \log u = 3\cos 2x \log 5x$

Diffn b/s w.r.t. x :

$\dfrac{1}{u}\dfrac{du}{dx} = -3 \times 2\sin 2x \log 5x + \dfrac{3\cos 2x}{\cancel{5}x} \times \cancel{5}$

$\Rightarrow \dfrac{du}{dx} = (5x)^{3\cos 2x} \times 3\left[\cos 2x - 2\sin 2x \log 5x\right]$

For $\dfrac{dv}{dx}$: $v = (x^2+2x)^{-1/2}$

$\Rightarrow \dfrac{dv}{dx} = -\dfrac{1}{2}(x^2+2x)^{-3/2}(2x+2)$

$= -(x+1)(x^2+2x)^{-3/2} = \dfrac{-(x+1)}{(x^2+2x)^{3/2}}$ Ans

$\therefore \dfrac{dy}{dx} = (5x)^{3\cos 2x} \times 3\left[\cos 2x - 2\sin 2x \log 5x\right] - \dfrac{(x+1)}{(x^2+2x)^{3/2}}$

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17. The given system of eqns can be expressed as: $AX = B$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ & $B = \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix}$;

$$|A| = \begin{vmatrix} 1 & 2 & 1 \\ 2 & 3 & 1 \\ 3 & 5 & 2 \end{vmatrix} = 1(6-5) - 2(4-3) + 1(10-9) = 1 - 2 + 1 = 0.$$

To find $\underline{\text{Adj } A}$

$$\text{Cofactor Matrix } A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix}$$

$$\therefore \text{Adj } A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \quad (\because \text{Adj } A = (\text{Cofactor Matrix})^T)$$

$$\therefore (\text{Adj } A) B = \begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -5 \\ 5 \end{bmatrix} \neq 0$$

Here $|A| = 0$ & $(\text{Adj } A) B \neq 0$

$\Rightarrow$ given system of eqns is inconsistent ~~with~~

i.e No Solution exists Ans.

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OR

$$BA = \begin{bmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

or $BA = 6I$

Given system of eqns can be expressed as:

$$\begin{pmatrix} 1 & -1 & 0 \\ 2 & 3 & 4 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$AX = \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix} \quad \text{where}$$

$$\Rightarrow X = A^{-1} \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$\Rightarrow X = \frac{1}{6} B \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 2 & 2 & -4 \\ -4 & 2 & -4 \\ 2 & -1 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ 17 \\ 7 \end{pmatrix}$$

$$\Rightarrow X = \frac{1}{6} \begin{pmatrix} 12 \\ -6 \\ 24 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 4 \end{pmatrix} \text{ or } x = 2, y = -1 \text{ \& } z = 4 \text{ Ans.}$$

18) we know $A = IA$

$$\Rightarrow \begin{pmatrix} 1 & -1 & 1 \\ 1 & -2 & -2 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} A$$

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$R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & -3 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} A$$

$R_2 \to (-) R_2$

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ -2 & 0 & 1 \end{pmatrix} A$$

$R_1 \to R_1 + R_2$ & $R_3 \to R_3 - 3R_2$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & -8 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 0 \\ -5 & 3 & 1 \end{pmatrix} A$$

$R_3 \to -\frac{1}{8} R_3$

$$\begin{pmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & -1 & 0 \\ 1 & -1 & 0 \\ 5/8 & -3/8 & -1/8 \end{pmatrix} A$$

$R_1 \to R_1 - 4R_3$ & $R_2 \to R_2 - 3R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ -7/8 & 1/8 & 3/8 \\ 5/8 & -3/8 & -1/8 \end{pmatrix} A$$

$\Rightarrow I = A^{-1}A$

$$\Rightarrow A^{-1} = \begin{pmatrix} -1/2 & 1/2 & 1/2 \\ -7/8 & 1/8 & 3/8 \\ 5/8 & -3/8 & -1/8 \end{pmatrix}$$ Ans.

19) $R_1 = \{(a,b) : 1+ab > 0;\ a,b \in R\}$.

(i) Reflexivity: $(a,a) \in R \because 1+a^2 > 0 \quad \forall\, a \in R$

$\Rightarrow R_1$ is reflexive

(ii) Symmetry: Let $(a,b) \in R_1 \Rightarrow 1+ab > 0$; $a,b \in R$

$\Rightarrow 1+ba > 0$

$\Rightarrow (b,a) \in R_1$

$\therefore R_1$ is symmetric.

(iii) Transitivity: Let $(a,b) \in R_1$ & $(b,c) \in R_1$; $a,b,c \in R$

$\Rightarrow 1+ab > 0$ & $1+bc > 0$

$\not\Rightarrow 1+ca > 0$.

e.g. $\left(\frac{-1}{2}, 1\right) \in R_1$ & $(1,5) \in R_1$

but $\left(\frac{-1}{2}, 5\right) \notin R_1$

$\therefore R_1$ is not transitive.

$\Rightarrow R_1$ is reflexive & symmetric but not transitive. Ans.

20) Given curve is : $y^2 = ax^3 + b$ — (1)

Diffn b/s of (1) w.r.t x :

$2y\frac{dy}{dx} = 3ax^2$ or $\frac{dy}{dx} = \frac{3ax^2}{2y}$.

$\Rightarrow m_T = \frac{dy}{dx}\Big]_{(2,3)} = \frac{\cancel{3}a(2)^{\cancel{2}}}{\cancel{2}\times\cancel{3}} = 2a$

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given tangent to curve (1) is :

$$y = 4x - 5 \quad \text{— (2)}$$

$$\Rightarrow m_T = 4$$

$$\therefore 2a = 4 \Rightarrow a = 2.$$

(2,3) lies on curve (1)

$$\Rightarrow 3^2 = 2(2)^3 + b$$

$$\Rightarrow 9 - 16 = b$$

$$\Rightarrow b = -7$$

$$\therefore \quad a = 2 \; \& \; b = -7 \quad \text{Ans.}$$

21) $f'(x) = \dfrac{2}{x-2} - 2x + 4 \quad , \quad x > 2$

$$\Rightarrow f'(x) = \frac{2 - 2x(x-2) + 4(x-2)}{x-2}$$

$$= \frac{2 - 2x^2 + 4x + 4x - 8}{x-2}$$

$$= \frac{-2x^2 + 8x - 6}{x-2}$$

$$= \frac{-2(x^2 - 4x + 3)}{x-2}$$

$$= \frac{-2(x-1)(x-3)}{(x-2)}$$

$f'(x) = 0 \Rightarrow \quad x = 1, 3.$

Here $x \neq 1 \quad \because x > 2.$

$\therefore$ Only turning point is $x = 3.$

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Interval	Sign of $f'(x)$	St. Inc / St. Dec.
$(2, 3)$	$+$	St. Inc.
$(3, \infty)$	$-$	St. dec.

$\therefore$ given func is st. inc on $(2,3)$ & st. decreasing on $(3, \infty)$ Ans.

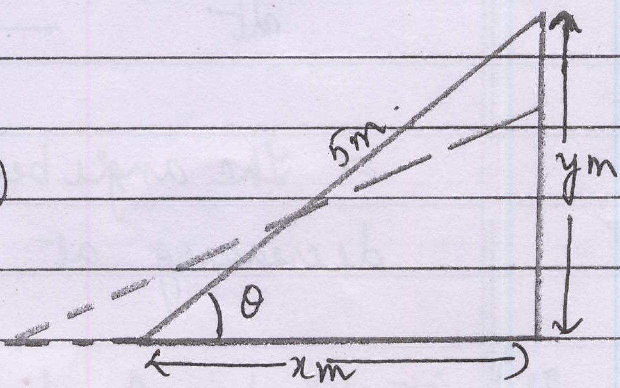
OR.

From the fig:

$$x^2 + y^2 = 5^2 \quad —(1)$$

$$\frac{dy}{dt} = -10 \text{ cm/s.} \quad —(2)$$
$$= \frac{-10}{100} \text{ m/s.}$$

When $x = 2m$, $\frac{d\theta}{dt} = ?$



From the fig: $\sin\theta = \frac{y}{5}$

$$\Rightarrow \cos\theta \frac{d\theta}{dt} = \frac{1}{5}\frac{dy}{dt}.$$

$$\Rightarrow \frac{d\theta}{dt} = \frac{\frac{1}{5}\left(\frac{-10}{100}\right)}{\cos\theta} = \frac{-1/50}{\frac{x}{5}}$$

$$\therefore \left.\frac{d\theta}{dt}\right]_{x=2m} = \frac{-1}{\cancel{50}_{10}} \times \frac{\cancel{5}}{2} = \frac{-1}{20} \text{ rad/s}$$

Ans.

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22) Let $4x-1 = A\frac{d}{dx}(x^2-4x+5) + B$

$\Rightarrow 4x-1 = A(2x-4) + B$

$\Rightarrow 2A = 4$ & $-1 = -4A+B$

$\Rightarrow A = 2$ & $-1+4(2) = B$

or $B = 7$

$\therefore \int \frac{4x-1}{x^2-4x+5}dx = 2\int \frac{2x-4}{x^2-4x+5}dx + 7\int \frac{dx}{x^2-4x+5}$

Let $x^2-4x+5 = t$

$\Rightarrow (2x-4)dx = dt$

$\Rightarrow \int \frac{4x-1}{x^2-4x+5}dx = 2\int \frac{dt}{t} + 7\int \frac{dx}{x^2-4x+4-4+5}$

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$$\Rightarrow \int \frac{4x-1}{x^2-4x+5}\,dx = 2\log|t| + 7\int \frac{dx}{(x-2)^2+(1)^2}$$

$$= 2\log|x^2-4x+5| + 7\tan^{-1}(x-2) + C$$

Ans.

23) Let $I = \int \frac{\sin x - \cos x}{1 - \sin x \cos x}\,dx$

Let $\cos x + \sin x = t$

$\Rightarrow (-\sin x + \cos x)\,dx = dt$

$\Rightarrow -(\sin x - \cos x)\,dx = dt$

Also $(\cos x + \sin x)^2 = t^2$

$\Rightarrow 1 + \sin 2x = t^2$

$\Rightarrow \sin 2x = t^2 - 1$

or $\sin x \cos x = \frac{t^2-1}{2}$.

$\therefore I = -\int \frac{dt}{1 - \frac{t^2-1}{2}} = -\int \frac{2\,dt}{2 - t^2 + 1}$

or $I = -2\int \frac{dt}{(\sqrt{3})^2 - t^2} = -\cancel{2} \times \frac{1}{\cancel{2}\sqrt{3}} \log\left|\frac{\sqrt{3}+t}{\sqrt{3}-t}\right| + C$

where $t = \sin x + \cos x$. Ans

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OR

Let $I = \int \frac{(x^2+3)/x^2 \, dx}{(x^4 - x^2 + 9)/x^2}$

$\Rightarrow I = \int \frac{1 + 3/x^2}{x^2 - 1 + 9/x^2} dx$

Let $x - \frac{3}{x} = t$

$\Rightarrow \left(1 + 3/x^2\right) dx = dt$

and $\left(x - \frac{3}{x}\right)^2 = t^2$

$\Rightarrow x^2 + \frac{9}{x^2} - 6 = t^2 \Rightarrow x^2 + \frac{9}{x^2} = t^2 + 6.$

$\therefore I = \int \frac{dt}{t^2+6-1}$

$\Rightarrow I = \int \frac{dt}{t^2+5}$

$\Rightarrow I = \int \frac{dt}{t^2 + (\sqrt{5})^2}$

or $I = \frac{1}{\sqrt{5}} \tan^{-1}\left(\frac{t}{\sqrt{5}}\right) + C$ where $t = x - \frac{3}{x}$ Ans.

—— x —— x —— x —— x ——

Section D

24) (i) Commutativity : Given: $(a,b) * (c,d) = (a+c, b+d)$; $a,b,c,d \in R$

$= (c+a, d+b)$

$= (c,d) * (a,b)$

$\Rightarrow$ $*$ on A is commutative

(ii) Associativity:

Here $((a,b)*(c,d))*(e,f) = (a+c, b+d)*(e,f)$; $a,b,c,d,e,f \in R$

$= (a+c+e, b+d+f)$ — ①

& $(a,b) * ((c,d) * (e,f)) = (a,b) * (c+e, d+f)$

$= (a+c+e, b+d+f)$ — ②

From ① & ② :

$((a,b)*(c,d))*(e,f) = (a,b) * ((b,d)*(e,f))$

$\Rightarrow$ $*$ on A is associative.

(iii) Let (e,e') be the identity element for $*$ on A

$\Rightarrow (a,b)*(e,e') = (a,b) \quad \forall (a,b) \in A$.

$= (e,e')*(a,b)$ where $a,b,e,e' \in R$

$\Rightarrow (a+e, b+e') = (a,b)$

$\Rightarrow \quad a+e = a$ & $b+e' = b$

$\Rightarrow \quad e = 0$ & $e' = 0$.

$\Rightarrow \quad (0,0)$ is the identity element for $*$ on A

(iv) Let $(x,y) \in A$ be the inverse of $(1,-2) \in A$ where $x,y \in R$

$\Rightarrow (x,y) * (1,-2) = (0,0) = (1,-2)*(x,y)$

$\Rightarrow \quad (x+1, y-2) = (0,0)$

$\Rightarrow \quad x+1 = 0$ & $y-2 = 0$

$\Rightarrow \quad x = -1$ & $y = 2$

$\therefore$ Inv of $(1,-2)$ in A is $(-1,2)$.

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25) Given: $f: R - \{-\frac{4}{3}\} \rightarrow R - \{\frac{4}{3}\}$ s.t.

$f(x) = \frac{4x+3}{3x+4}$

For One-One: Let $f(x_1) = f(x_2)$; $x_1, x_2 \in R - \{-\frac{4}{3}\}$

$\Rightarrow \frac{4x_1+3}{3x_1+4} = \frac{4x_2+3}{3x_2+4}$

$\Rightarrow \cancel{12x_1x_2} + 16x_1 + 9x_2 + \cancel{12} = \cancel{12x_2x_1} + 9x_1 + 16x_2 + \cancel{12}$

$\Rightarrow 7x_1 = 7x_2$

or $x_1 = x_2$

$\Rightarrow f$ is one-one.

For Onto Let $y \in R - \{\frac{4}{3}\}$

$\Rightarrow y = f(x)$

$\Rightarrow y = \frac{4x+3}{3x+4}$

$\Rightarrow 3xy + 4y = 4x + 3$

$\Rightarrow 3xy - 4x = 3 - 4y$

or $x(3y-4) = 3 - 4y$

or $x = \frac{3-4y}{3y-4} \in R - \{-\frac{4}{3}\}$

$\left(\because \text{if } x = -4/3 \right.$

$\Rightarrow -4/3 = \frac{3-4y}{3y-4}$

$\Rightarrow \cancel{-4y} + \frac{16}{3} = 3 - \cancel{4y}$

which is absurd $\left. \right)$

$\therefore \forall$ element in the co-domain

$\exists$ a pre-image in the domain

$\Rightarrow f$ is onto.

$\because f$ is one-one, onto, it is a bijection

$\Rightarrow f^{-1}$ exists and is defined as:

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$$f^{-1} : R - \left\{\frac{4}{3}\right\} \longrightarrow R - \left\{\frac{-4}{3}\right\}$$

$$\text{s.t. } f^{-1}(y) = \frac{3-4y}{3y-4}$$

$$\therefore f^{-1}(x) = \frac{3-4x}{3x-4}$$

If $f^{-1}(x) = 2$

$$\Rightarrow \frac{3-4x}{3x-4} = 2$$

$$\Rightarrow 3-4x = 6x-8$$

or $\quad 11 = 10x$

$$\Rightarrow \boxed{x = \frac{11}{10}}$$

OR.

$f : A \to B \quad$ s.t. $f(x) = x^2 - x \quad , x \in A$

& $g : A \to B \quad$ s.t. $g(x) = 2\left|x - \frac{1}{2}\right| - 1 \quad , x \in A$

$$\therefore g\circ f : A \to B \text{ s.t. } g\circ f(x) = g(f(x))$$

$$= g(x^2 - x)$$

$$= 2\left|x^2 - x - \frac{1}{2}\right| - 1 \quad , x \in A$$

Here $f = \{(-1,2), (0,0), (1,0), (2,2)\}$

and $g = \{(-1,2), (0,0), (1,0), (2,2)\}$

$\therefore g\circ f(-1) = g(f(-1)) = g(2) = 2$

$g\circ f(0) = g(f(0)) = g(0) = 0$

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$gof(1) = g(f(1)) = g(0) = 0$
$gof(2) = g(f(2)) = g(2) = 2$

$\therefore gof = \{(-1,2), (0,0), (1,0), (2,2)\}$
$= g = f$

Hence $g = f = gof$.
gof is not invertible because it is neither one-one not onto.

26) Let $I = \int_2^4 (6x^3 + 5x - 7)\,dx$

Here $f(x) = 6x^3 + 5x - 7$
$nh = 4 - 2 = 2.$

We know: $\int_2^4 (6x^3+5x-7) = \lim_{h \to 0} h\left[f(2) + f(2+h) + f(2+2h) + \cdots + f(2+(n-1)h)\right].$

$\Rightarrow I = \lim_{h \to 0} h\left[(51) + (51 + 6h^3 + 36h^2 + 77h) + (51 + 48h^3 + 144h^2 + 154h) + \cdots + (51 + 6(n-1)^3h^3 + 36(n-1)^2h^2 + 77(n-1)h)\right]$

$\Rightarrow I = \lim_{h \to 0} h\left[51n + 6h^3(1 + 8 + \cdots + (n-1)^3) + 36h^2(1 + 4 + \cdots + (n-1)^2) + 77h(1 + 2 + \cdots + (n-1))\right]$

$\Rightarrow I = \lim_{h \to 0} h\left[51n + 6h^3\left(\frac{n(n-1)}{2}\right)^2 + \frac{36h^2\, n(n-1)(2n-1)}{6} + 77h\,\frac{n(n-1)}{2}\right]$

$$\Rightarrow I = \lim_{h\to 0}\left[51nh + \frac{6}{4}\left(nh(nh-h)\right)^2 + 6nh(nh-h)(2nh-h) + \frac{77nh(nh-h)}{2}\right]$$

$$\Rightarrow I = 51\times 2 + \frac{3}{2}\left(2(2-0)\right)^2 + 6(2)(2-0)(4-0) + \frac{77\times\cancel{2}(2-0)}{\cancel{2}}$$

$$\Rightarrow I = 102 + \frac{3}{\cancel{2}}\times\cancel{16}^{8} + 96 + 154$$

$$\Rightarrow I = 376 \quad \text{Ans.}$$

OR.

$$\text{Let } I = \int_0^{\pi} \underset{2^{nd}}{e^{2x}}\, \underset{1^{st}}{\sin\left(\frac{\pi}{4}+x\right)}\, dx \quad -①$$

$$\Rightarrow I = \left[\sin\left(\frac{\pi}{4}+x\right)\times\left(\frac{e^{2x}}{2}\right) - \int\left(\cos\left(\frac{\pi}{4}+x\right)\times\frac{e^{2x}}{2}\right)dx\right]_0^{\pi}$$

$$\Rightarrow I = \left[\frac{e^{2x}}{2}\sin\left(\frac{\pi}{4}+x\right)\right]_0^{\pi} - \frac{1}{2}\left\{\cos\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{2} + \int\sin\left(\frac{\pi}{4}+x\right)\times\frac{e^{2x}}{2}dx\right\}_0^{\pi}$$

$$\Rightarrow I = \left[\frac{e^{2\pi}}{2}\sin\left(\frac{5\pi}{4}\right) - \frac{1}{2}\sin\frac{\pi}{4}\right] - \frac{1}{2}\left\{\cos\left(\frac{\pi}{4}+x\right)\frac{e^{2x}}{2}\right\}_0^{\pi} - \frac{1}{4}I$$

$$\Rightarrow \frac{5}{4}I = -\frac{1}{2\sqrt{2}}e^{2\pi} - \frac{1}{2\sqrt{2}} - \frac{1}{2}\left\{\frac{e^{2\pi}}{2}\left(\frac{-1}{\sqrt{2}}\right) - \frac{1}{2}\times\frac{1}{\sqrt{2}}\right\}$$

$$\Rightarrow \frac{5}{4}I = \left(\frac{-1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}\right)e^{2\pi} - \frac{1}{2\sqrt{2}} + \frac{1}{4\sqrt{2}}$$

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$$\Rightarrow \frac{5}{4}I = \frac{-1}{4\sqrt{2}}e^{2\pi} - \frac{1}{4\sqrt{2}}$$

$$\Rightarrow 5I = -\frac{e^{2\pi}}{\sqrt{2}} - \frac{1}{\sqrt{2}}$$

$$\text{or } I = \frac{-1}{5}\left(\frac{e^{2\pi}+1}{\sqrt{2}}\right) \text{ Ans.}$$

27) Let $I = \int_{-\pi}^{\pi} \frac{x\sin x}{1+3\sin^2 x}\,dx$.

Here $f(x) = \frac{x\sin x}{1+3\sin^2 x}$

$$\therefore f(-x) = \frac{-x\sin(-x)}{1+3\sin^2(-x)} = \frac{x\sin x}{1+3\sin^2 x} = f(x)$$

$$\therefore I = 2\int_0^{\pi} \frac{x\sin x}{1+3\sin^2 x}\,dx \text{ — } 1$$

$$\Rightarrow I = 2\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+3\sin^2(\pi-x)}\,dx$$

$$\Rightarrow I = 2\int_0^{\pi} \frac{(\pi-x)\sin x}{1+3\sin^2 x}\,dx \text{ — } 2$$

1 + 2 gives :

$$2I = 2\pi\int_0^{\pi} \frac{\sin x}{1+3\sin^2 x}\,dx$$

$$\Rightarrow I = \pi\int_0^{\pi} \frac{\sin x}{1+3\sin^2 x}\,dx$$

$$\Rightarrow I = \pi \int_0^{\pi} \frac{\sin x}{1+3(1-\cos^2 x)}\, dx$$

$$\Rightarrow I = \pi \int_0^{\pi} \frac{\sin x}{4-3\cos^2 x}\, dx$$

Let $\cos x = t \Rightarrow -\sin x\, dx = dt$

When $x = 0,\ t = 1$

$x = \pi,\ t = -1$

$$\therefore I = -\pi \int_1^{-1} \frac{dt}{4-3t^2}$$

$$\Rightarrow I = \pi \int_{-1}^{1} \frac{dt}{4-3t^2} = \pi \int_{-1}^{1} \frac{dt}{(2)^2-(\sqrt{3}t)^2}$$

$$\Rightarrow I = \pi \left[\frac{1}{2(2)\sqrt{3}} \log \left| \frac{2+\sqrt{3}t}{2-\sqrt{3}t} \right| \right]_{-1}^{1}$$

$$\Rightarrow I = \frac{\pi}{4\sqrt{3}} \left[\log \left| \frac{2+\sqrt{3}}{2-\sqrt{3}} \right| - \log \left| \frac{2-\sqrt{3}}{2+\sqrt{3}} \right| \right]$$

$$\Rightarrow I = \frac{\pi}{4\sqrt{3}} \left[\log \left| \frac{2+\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2-\sqrt{3}} \right| \right]$$

$$\Rightarrow I = \frac{\pi}{4\sqrt{3}} \left[\log \frac{(2+\sqrt{3})^2}{(2-\sqrt{3})^2} \right]$$

$$\Rightarrow I = \frac{\cancel{2}\pi}{2\cancel{4}\sqrt{3}} \log \left| \frac{2+\sqrt{3}}{2-\sqrt{3}} \right| \text{ Ans.}$$

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28) Given:

$P_{sq} + P_{circle} = k$ — ①

$\Rightarrow 4x + 2\pi r = k$ — ②

where $x \to$ side of square

& $r \to$ radius of circle.

$\therefore A = x^2 + \pi r^2$ — ③

sum of areas of sq & circle

$\Rightarrow A = x^2 + \pi\left(\frac{k-4x}{2\pi}\right)^2$ — ④

$\Rightarrow \frac{dA}{dx} = 2x + \cancel{2}\pi\left(\frac{k-4x}{2\cancel{\pi}}\right) \times \left(\frac{-4^2}{2\pi}\right)$

$\Rightarrow \frac{dA}{dx} = 2x - \frac{2}{\pi}(k-4x)$

$\frac{dA}{dx} = 0$ for least A

$\Rightarrow \cancel{2}x = \frac{\cancel{2}}{\pi}(k-4x)$

$\Rightarrow \pi x = k - 4x$

or $(\pi+4)x = k$ or $x = \frac{k}{\pi+4}$.

$\therefore \frac{d^2A}{dx^2} = 2 - \frac{2}{\pi}(-4) = 2 + \frac{8}{\pi} > 0$ for $x = \frac{k}{\pi+4}$

$\Rightarrow$ A is least when $x = \frac{k}{\pi+4}$

& when $x = \frac{k}{\pi+4}$, $2r = \frac{k-4x}{\pi} = \frac{k - 4\left(\frac{k}{\pi+4}\right)}{\pi}$

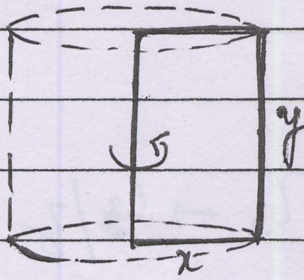
or $2r = \frac{\pi k + \cancel{4k} - \cancel{4k}}{\pi(\pi+4)} = \frac{k}{\pi+4} = x$

i.e when A is least $2r = x$

or diameter of circle = side of square

Hence Proved.

OR



Let V denote the volume of the cylinder formed by revolving a rectangle of length x and breadth y about its breadth.

$\therefore V = \pi x^2 y$ — (1)

$P = 2(x+y) = 36$ (given)

$\Rightarrow x + y = 18$ — (2)

$\therefore V = \pi x^2 (18 - x)$

or $V = 18\pi x^2 - \pi x^3$ — (3)

$\therefore \frac{dV}{dx} = 36\pi x - 3\pi x^2$

For max^m V, $\frac{dV}{dx} = 0$

$\Rightarrow 12\cancel{36}\cancel{\pi}x = \cancel{3}\cancel{\pi}x^2$

$\Rightarrow 12x - x^2 = 0$

$\Rightarrow x(12 - x) = 0$

$\Rightarrow x = 0, 12$

↳ not possible

$\Rightarrow x = 12$

$\therefore \frac{d^2V}{dx^2} = 36\pi - 6\pi x$

$\therefore \left.\frac{d^2V}{dx^2}\right]_{x=12} < 0 \Rightarrow$ V is max^m when $x = 12$

When $x = 12$, $y = 18 - 12 = 6$.

$\therefore$ Reqd dimensions are: 12 cm, 6 cm.
& Max^m volume $= \pi x^2 y = \pi(12)^2 \times 6 = 864\pi$ cm^3 Ans.

29) $\begin{vmatrix} p & q-y & r-z \\ p-x & q & r-z \\ p-x & q-y & r \end{vmatrix} = 0.$

$C_1 \to C_1/x$, $C_2 \to C_2/y$, $C_3 \to C_3/z$

$\therefore \Delta = \begin{vmatrix} p/x & q/y - 1 & r/z - 1 \\ p/x - 1 & q/y & r/z - 1 \\ p/x - 1 & q/y - 1 & r/z \end{vmatrix} = 0.$

$C_1 \to C_1 + C_2 + C_3$.

$\Rightarrow \begin{vmatrix} \frac{p}{x} + \frac{q}{y} + \frac{r}{z} - 2 & \frac{q}{y} - 1 & r/z - 1 \\ \frac{p}{x} + \frac{q}{y} + \frac{r}{z} - 2 & q/y & r/z - 1 \\ \frac{p}{x} + \frac{q}{y} + \frac{r}{z} - 2 & q/y - 1 & r/z \end{vmatrix} = 0$

Taking $\left(\frac{p}{x} + \frac{q}{y} + \frac{r}{z} - 2\right)$ common from C_1 we get:

$\left(\frac{p}{x} + \frac{q}{y} + \frac{r}{z} - 2\right) \begin{vmatrix} 1 & q/y - 1 & \frac{r}{z} - 1 \\ 1 & q/y & \frac{r}{z} - 1 \\ 1 & q/y - 1 & \frac{r}{z} \end{vmatrix} = 0$

$R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

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$$\left(\frac{p}{x}+\frac{q}{y}+\frac{r}{z}-2\right)\begin{vmatrix} 1 & \frac{q}{y}-1 & \frac{r}{z}-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=0$$

Exp. along C_1, we get

$$\left(\frac{p}{x}+\frac{q}{y}+\frac{r}{z}-2\right)(1)=0$$

$$\Rightarrow \frac{p}{x}+\frac{q}{y}+\frac{r}{z}-2=0$$

$$\Rightarrow \frac{p}{x}+\frac{q}{y}+\frac{r}{z}=2 \text{ Ans.}$$